October Math Gems

Problem of the week 3

§1 problems

Problem 1.1. The length of the perimeter of a right triangle is 60 inches and the length of the altitude perpendicular to the hypotenuse is 12 inches. Find the sides of the triangle.

Solution. Let a, b, and c denote the sides, the last being the hypotenuse. The three parts of the condition are expressed by

$$\begin{aligned} a+b+c &= 60 \\ a^2+b^2 &= c^2 \\ ab &= 12\text{c}. \end{aligned}$$

Observing that

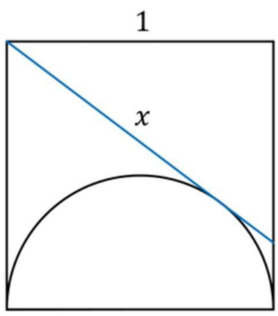
$$(a+b)^2 = a^2+b^2+2ab$$

we obtain

$$(60-c)^2 = c^2+24c.$$

Hence $c = 25$ and either $a = 15, b = 20$ or $a = 20, b = 15$ (no difference for the triangle). □

Problem 1.2. A square with the side length 1 has an inscribed semicircle along one of its sides as shown here, Find the value of x.



Solution. We will use the principle of the tangents of the circle are equal.

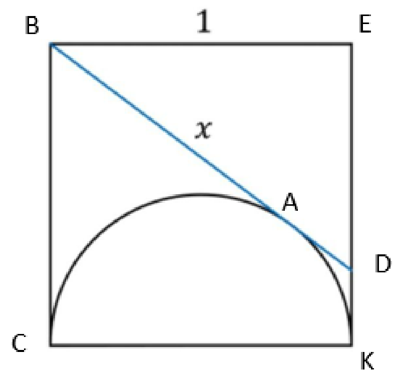
$$BA = BC = 1 \qquad \text{and} \qquad AD = DK$$

$$(ED)^2 + (BE)^2 = (BA+AD)^2 = x^2 \implies 1 + (1-AD)^2 = (1+AD)^2$$

So,

$$AD = \frac{1}{4} \implies x = 1 + \frac{1}{4} = \frac{5}{4}$$

□



Problem 1.3. If $x + \frac{1}{16x} = 1$, then the value of

$$64x^3 + \frac{1}{64x^3}$$

is?

Solution.

$$x + \frac{1}{16x} = 1$$

$$x^3 + \frac{1}{(16)^3x^3} = (x + \frac{1}{16x})(x^2 - \frac{1}{16} + \frac{1}{(16)^2x^2})$$

$$(x + \frac{1}{16x})^2 = (x^2 + \frac{1}{(16)^2x^2} + \frac{1}{8}) \implies x^2 + \frac{1}{(16)^2x^2} = 1 - \frac{1}{8} = \frac{7}{8}$$

So,

$$x^3 + \frac{1}{(16)^3x^3} = 1 \times (\frac{7}{8} - \frac{1}{16}) = \frac{13}{16}$$

As required the value of $64x^3 + \frac{1}{64x^3}$,

$$64 \times (x^3 + \frac{1}{(16)^3x^3}) = 64 \times \frac{13}{16} = 52$$

□

Problem 1.4. If $f(x) = ax^2 + bx + c$, and $f(2) = 0$, $f(3) = 11$, then find the value of

$$a - b - c =$$

Solution.

$$f(2) = 4a + 2b + c = 0$$

$$f(3) = 9a + 3b + c = 11$$

from the value of c

$$\begin{cases} c = -2b - 4a \\ c = 11 - 9a - 3b \end{cases}$$

So we get that

$$-4a - 2b = -9a - 3b + 11 \implies 5a + b = 11$$

$$b = 11 - 5a \qquad c = 6a - 22$$

$$a - b - c = a - (11 - 5a) - (6a - 22) = -11 + 22 = 11$$

□

Problem 1.5. If

$$a^4 + a^2b^2 + b^4 = 8$$

and

$$a^2 + ab + b^2 = 4$$

Then compute the value of

$$ab$$

?

Solution.

$$a^2 + ab + b^2 = 4 \implies a^2 + b^2 = 4 - ab$$
$$(a^2 + b^2)^2 = (4 - ab)^2 \implies a^4 + b^4 = 16 - 8ab - a^2b^2$$

Form

$$a^4 + a^2b^2 + b^4 = 8 \implies a^4 + b^4 = 8 - a^2b^2$$

So,

$$8 - a^2b^2 = 16 - 8ab - a^2b^2 \implies ab = 1$$

$\square$

Problem 1.6. If $x^2 + y^2 + z^2 = xy + yz + zx$, then compute the value of

$$\frac{4x + 2y - 3z}{2x}$$

Solution.

$$x^2 + y^2 + z^2 - xy - zy - zx = 0$$
$$2(x^2 + y^2 + z^2 - xy - zy - zx) = 0$$
$$(x - y)^2 + (x - z)^2 + (y - a)^2 = 0$$

From this we get that

$$x = y = z$$

So,

$$\frac{4x + 2y - 3z}{2x} = \frac{3x}{2x} = \frac{3}{2}$$

$\square$

Problem 1.7. Solve in R

$$\sqrt{5 + x^2} + \sqrt{5 - x^2} = \frac{4}{\sqrt{5 - x^2}}$$

Solution.

$$\sqrt{5 - x^2} \times (\sqrt{5 + x^2} + \sqrt{5 - x^2}) = 4$$
$$\sqrt{(5 - x^2)(5 + x^2)} + 5 - x^2 = 4$$
$$\sqrt{(5 - x^2)(5 + x^2)} = x^2 - 1$$

squaring both sides

$$25 - x^4 = x^4 - 2x^2 + 1$$
$$2x^4 - 2x^2 - 24 = 0 \implies x = \pm 2$$

$\square$

Problem 1.8. Find the value of mn if

$$2^m - 2^n = 2016$$

Solution. Suppose that

$$m = n + k$$

So,

$$2^{n+k} - 2^n = 2016 \implies 2^n(2^k - 1) = 2016$$

$$2^n(2^k - 1) = 2^5 \times 63 \implies 2^n = 2^5 \qquad \text{and} \qquad (2^k - 1) = 63$$

So,

$$n = 5 \qquad \text{and} \qquad k = 6 \qquad \text{and} \qquad m = 11$$

□

Problem 1.9. From the following equation

$$1 + \sqrt{3^x} = 2^x$$

Find x.

Solution. □

Problem 1.10. If $(x - a)(x - b) = 1$ and $a - b + 5 = 0$, then compute the value of

$$(x - a)^3 - \frac{1}{(x - a)^3}$$

Solution.

$$a - b + 5 = 0 \implies b = a + 5$$

$$(x - a)(x - b) = 1 \implies (x - a)((x - a) - 5) = 1$$

We can say that

$$(x - a)^2 + 5(x - a) = 1 \implies (x - a) - \frac{1}{(x - a)} = 5 \to (1)$$

We will use the identity

$$a^3 - b^3 = (a - b)^3 + 3ab(a - b)$$

So,

$$(x - a)^3 - \frac{1}{(x - a)^3} = ((x - a) - \frac{1}{(x - a)})^3 + 3((x - a) - \frac{1}{(x - a)})$$

$$5^3 + 3 \times 5 = 125 + 15 = 140$$

□

Problem 1.11. Solve the following equation

$$\frac{1}{x} - 4\sqrt{\frac{x + 1}{x}} + 5 = 0$$

Solution.

$$\frac{1}{x} - 4\sqrt{1 + \frac{1}{x}} + 5 = 0$$

$$(\frac{1}{x} + 5)^2 = (4\sqrt{1 + \frac{1}{x}})^2$$

$$\frac{1}{x^2} + 25 + \frac{10}{x} = 16 + \frac{16}{x}$$

$$\frac{1}{x^2} + 9 - \frac{6}{x} = 0$$

Let $x = \frac{1}{y}$. So,

$$y^2 - 6y + 9 = 0 \implies y = 3$$

So, $x = \frac{1}{3}$. ☐

Problem 1.12. If 5 positive consecutive numbers have a sum of 10^{2018}, then what is the value of the middle number?

Solution. Let the five numbers be $n, n+1, n+2, n+3$ and $n+4$. Therefore,

$$5n + 10 = 10^{2018}$$

So, the value of n is $\frac{10^{2018}-10}{5}$. Substituting this value in $n+2$, we get

$$n + 2 = \frac{10^{2018}}{5} = 2(10^{2017})$$

☐

Problem 1.13. Find the value of x

$$5^x - 5^{x-1} = 500$$

Solution. First, we divide both sides by 5^x and get

$$\frac{4}{5} = \frac{500}{5^x}$$

Therefore,

$$5^x = 625$$

So, $x = 4$ ☐

Problem 1.14. In a room, there are 144 people. They are joined by n people who are carrying k coins each. When these coins are divided among the $n + 144$ people, each person had 2 coins. Find the minimum possible value of $2n + k$.

Solution. First, we can deduce that

$$n \times k = 2(n + 144)$$

Therefore,

$$n(k - 2) = 288$$

This expression is at its minimum when n is not too large. We can try with the first 20 integers that lead to a positive integer value for k. After trying, we get that the least value for n is 12. So, the minimum value of $2n + k$ is 50 ☐

Problem 1.15. Let x be a real number such that

$$(\sqrt{6})^x - 3^x = 2^{x-2}$$

Evaluate the following expression

$$\frac{4^{x+1}}{9^{x-1}}$$

Solution. First, let $u = (\sqrt{2})^x$ and $v = (\sqrt{3})^x$. Therefore, we can write the expression as

$$uv - v^2 = \frac{u^2}{4}$$

This can be factored to

$$(\frac{u}{2} - v) = 0$$

This implies that $\frac{u}{v} = 2$, so $(\frac{4}{9})^x = 2^4$ The requested expression is $\frac{4^{x+1}}{9^{x-1}}$, and it can be written as

$$4 \times 9 \times \left(\frac{4}{9}\right)^x = 36 \times 16 = 576$$

□

Problem 1.16. There are non-zero real numbers a and b so that the roots of $x^2 + ax + b$ are $3a$ and $3b$. There are relatively prime positive integers m and n such that $a - b = \frac{m}{n}$. Find $m + n$

Solution. First, we can see that $3a + 3b = -a$ and $3a \times 3b = b$ (Using the sum and product formula for the roots of a quadratic function). Therefore, we can deduce that

$$a = \frac{1}{9}$$

$$b = \frac{-4}{27}$$

Therefore,

$$a - b = \frac{1}{9} - \frac{-4}{27} = \frac{7}{27}$$

So, the value of $m + n = 7 + 27 = 34$ □

Problem 1.17. The sum of the ages of a father and his son is 55. The age of the father is the reverse of the age of the son. What is the age of both the father and the son?

Solution. First, let us assume the age of the father is xy and the age of the son is yx. This can also be written as

$$xy = 10x + y$$

$$yx = 10y + x$$

Summing these two equations, we get $x + y = 5$. Since $x > y$ (Because the father is older), there are 3 possibilities

$$x = 5, y = 0$$

$$x = 4, y = 1$$

$$x = 3, y = 2$$

We can eliminate the first possibility since if the son's age is 5, its reverse would be 5 as well not 50. We can eliminate the third possibility since an age gap of 9 years between the father and the son is unrealistic. Therefore, the second possibility is true, where the father's age is 41 and the son's age is 14. □

Problem 1.18. Given that

$$\frac{x}{2x^2+5x+2} = \frac{1}{2}$$

Find the value of

$$x + \frac{1}{x}$$

Solution. First, we can clearly deduce that

$$2x^2+5x+2 = 2x$$

This simplifies to

$$2x^2+3x+2 = 0$$

By dividing both sides by $2x$, we get

$$x + \frac{1}{x} + \frac{3}{2} = 0$$

Therefore,

$$x + \frac{1}{x} = \frac{-3}{2}$$

□

Problem 1.19. Solve for real values of x

$$3^{2x+1} + 4(3^x) - 15 = 0$$

Solution. Let $u = 3^x$. Then, we can rewrite the expression as

$$3u^2 + 4u - 15 = 0$$

This quadratic function has the roots -3 and $\frac{5}{3}$. However, we can exclude the -3 since 3^x is positive for real values of x. Therefore, we can say that

$$3^x = \frac{5}{3}$$

$$x = \frac{ln(\frac{5}{3})}{ln(3)} \approx 0.465$$

□

Problem 1.20. Let a and b be positive real numbers satisfying

$$a - 12b = 11 - \frac{100}{a} \quad \text{and} \quad a - \frac{12}{b} = 4 - \frac{100}{a}.$$

Then $a + b = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

Solution. Subtracting the first equation from the second yields $12b - \frac{12}{b} = -7$ which simplifies to $12b^2+7b-12 = 0$. The quadratic factors to give $(3b+4)(4b-3) = 0$, so $b = \frac{3}{4}$. Substituting this into the first equation and simplifying gives $a^2 - 20a + 100 = 0$, which has solution $a = 10$. Thus, $a + b = 10 + \frac{3}{4} = \frac{43}{4}$. The requested sum is $43 + 4 = 47$. □
